SYDNEY GRAMMAR SCHOOL

2014 Assessment Examination

FORM VI
MATHEMATICS EXTENSION 1

Monday 19th May 2014

General Instructions
- Writing time — 1 hour 30 minutes
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 55 Marks
- All questions may be attempted.

Section I – 7 Marks
- Questions 1–7 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 48 Marks
- Questions 8–11 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection
- Write your candidate number on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Eight.
- Write your candidate number on this question paper and submit it with your answers.

Checklist
- SGS booklets — 4 per boy
- Multiple choice answer sheet
- Candidature — 123 boys

Examiner
LRP

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The exact value of $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right)\cos^{-1}\left(\frac{1}{2}\right)$ in radians is: **1**

(A) $0{\cdot}822$

(B) 2700

(C) $\frac{\pi^2}{12}$

(D) $\frac{\pi^2}{24}$

QUESTION TWO

If $x+\alpha$ is a factor of $7x^3+9x^2-5\alpha x$, where $\alpha \neq 0$, then the value of α is: **1**

(A) 2

(B) $\frac{4}{7}$

(C) $-\frac{4}{7}$

(D) -2

QUESTION THREE

A particle is moving in a straight line in such a way that its displacement, x metres, at time t seconds is given by $x = 2{\cdot}5t + 5\cos(0{\cdot}5t)$, where $t \geq 0$. **1**

The minimum velocity of the particle, in metres per second, is:

(A) -5

(B) $-2{\cdot}5$

(C) 0

(D) $2{\cdot}5$

Exam continues next page ...

QUESTION FOUR

If $f'(x) = \cos x$ and $f\left(\frac{\pi}{2}\right) = 0$, then $f(x)$ is equal to: **1**

(A) $\sin x$

(B) $1 - \sin x$

(C) $\sin x - 1$

(D) $\sin x + 1$

QUESTION FIVE

Which one of the following is **not** equal to $\tan\left(\frac{\pi}{5}\right)$? **1**

(A) $\dfrac{1}{\cot\left(\frac{\pi}{5}\right)}$

(B) $\cot\left(\frac{3\pi}{10}\right)$

(C) $\dfrac{2\tan\left(\frac{\pi}{10}\right)}{1-\tan^2\left(\frac{\pi}{10}\right)}$

(D) $\dfrac{2\tan\left(\frac{2\pi}{5}\right)}{1-\tan^2\left(\frac{2\pi}{5}\right)}$

QUESTION SIX

The velocity v of a particle as a function of its displacement x is given by $v = \dfrac{2}{\sqrt{1-x^2}}$. **1**

Its acceleration is:

(A) $2\sin^{-1} x$

(B) $\dfrac{4x}{(1-x^2)^2}$

(C) $\dfrac{2x}{(1-x^2)^2}$

(D) $\dfrac{2x}{(1-x^2)^{\frac{3}{2}}}$

Exam continues overleaf ...

QUESTION SEVEN

The graphs of $y = ax$ and $y = \tan^{-1}(bx)$ intersect exactly **three** times if: $\boxed{1}$

(A) $a = b$

(B) $a = -b$

(C) $0 < b < a$

(D) $0 < a < b$

End of Section I

Exam continues next page ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION EIGHT (12 marks) Use a separate writing booklet. **Marks**

(a) Let $f(x) = 2\cos^{-1}\left(\frac{x}{3}\right)$. What is the domain of $f(x)$? **1**

(b) Find the exact value of $\displaystyle\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx$. **2**

(c) Use the expansion of $\cos(A-B)$ to show that $\cos 15^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$. **2**

(d) Solve the equation $2\sin^2\theta = \sin 2\theta$ for $0 \le \theta \le 2\pi$. **3**

(e) A particle is moving in simple harmonic motion according to the equation

$$x = 2\cos\left(\frac{\pi}{10}t\right) + 3,$$

where x is the displacement in metres at time t seconds.

(i) Find the amplitude and period of the motion. **2**

(ii) Sketch the displacement-time graph over the first 60 seconds. **2**

Exam continues overleaf ...

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) The acceleration of a particle moving in a straight line is given by $\ddot{x} = 10x - 4x^3$, where x is the displacement in metres at time t seconds. Initially the particle is stationary at $x = \sqrt{6}$ metres.

(i) Find the initial acceleration. **1**

(ii) Find an expression for v^2 as a function of x. **2**

(b) Consider the polynomial $P(x) = x^4 - x^3 - 3x^2 + 5x - 2$.

(i) Show that 1 and -2 are zeroes of $P(x)$. **1**

(ii) Factorise $P(x)$ into linear factors. **3**

(iii) Without the aid of calculus, sketch the graph of $P(x)$, clearly indicating all intercepts with the axes. **2**

(c) (i) Differentiate $x \sin^{-1} x + \sqrt{1 - x^2}$. **2**

(ii) Hence evaluate $\displaystyle\int_0^1 \sin^{-1} x \, dx$. **1**

Exam continues next page ...

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a) Using the fact that $\cos 3x = 4\cos^3 x - 3\cos x$, find general solutions of the equation $\cos 3x + 2\cos x = 0$. $\boxed{4}$

(b) A particle is moving in simple harmonic motion according to $x = 2\sin 3t - 2\sqrt{3}\cos 3t$, where x is its displacement in metres from a fixed point O at time t seconds.

(i) Express x in the form $x = R\sin(3t - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{\pi}{2}$. $\boxed{2}$

(ii) Find the initial displacement and velocity. $\boxed{2}$

(iii) Find the first time that the particle is 2 metres from O. $\boxed{2}$

(c) A body is moving with velocity $v = 4 - 2t\,\text{ms}^{-1}$, where t is time in seconds. Find the total distance it travels in the first 5 seconds of its motion. $\boxed{2}$

QUESTION ELEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) If $\tan\alpha$ and $\tan\beta$ are two values of $\tan\theta$ which satisfy the quadratic equation $a\tan^2\theta + b\tan\theta + c = 0$:

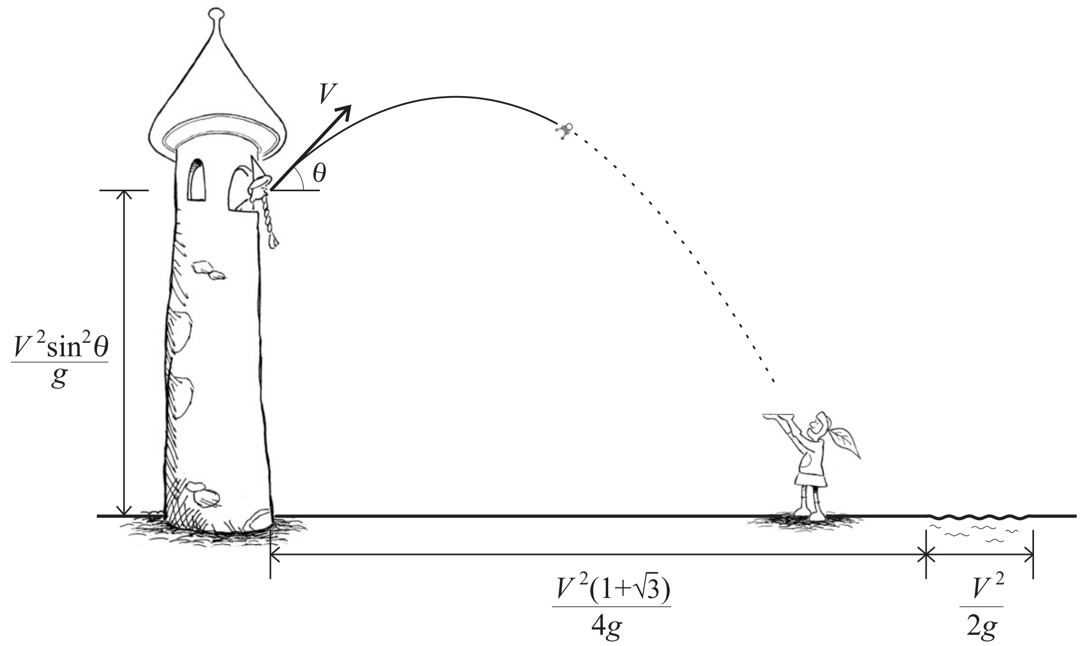
(i) Find $\tan(\alpha + \beta)$ in terms of a, b and c. $\boxed{2}$

(ii) Show that $\tan^2(\alpha - \beta) = \dfrac{b^2 - 4ac}{(a+c)^2}$. $\boxed{2}$

QUESTION ELEVEN CONTINUES ON THE NEXT PAGE

QUESTION ELEVEN (Continued)

(b)



Rapunzel is trapped on the top floor of an enchanted tower. She throws a set of keys that unlock the tower to a handsome prince standing on the ground below. The keys are projected with an initial velocity $V\,\text{ms}^{-1}$ and at an angle θ to the horizontal, where $0^\circ < \theta < 90^\circ$.

The point of projection is a window located $\dfrac{V^2\sin^2\theta}{g}$ metres above the ground, where g is the acceleration due to gravity. Assume that the prince can't catch and the keys fall to the ground. The horizontal and vertical displacement equations of the keys at time t are given by

$$x = Vt\cos\theta \quad \text{and} \quad y = Vt\sin\theta - \frac{gt^2}{2} + \frac{V^2\sin^2\theta}{g}.$$

(i) Show that the Cartesian equation of the path of the keys is given by **[1]**

$$y = x\tan\theta - \frac{gx^2\sec^2\theta}{2V^2} + \frac{V^2\sin^2\theta}{g}.$$

(ii) Show that the horizontal range of the keys is given by $\dfrac{V^2\left(1+\sqrt{3}\right)\sin 2\theta}{2g}$ metres. **[3]**

(iii) The near edge of a moat lies $\dfrac{V^2\left(1+\sqrt{3}\right)}{4g}$ metres from the base of the tower. **[4]**

The moat is $\dfrac{V^2}{2g}$ metres wide. Find the values of θ for which the keys will land either side of the moat.

End of Section II

END OF EXAMINATION

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The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}\,x^{n+1},\;\; n \neq -1;\;\; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\;\; x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}\,e^{ax},\;\; a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\;\; a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\;\; a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\;\; a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\;\; a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\;\; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\;\; a > 0,\;\; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\;\; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

$$\text{NOTE}: \;\ln x = \log_e x,\; x > 0$$

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

Candidate number: ....

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

2014 Form VI May Assessment

SOLUTIONS

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* MULTIPLE CHOICE

Q1 C
Q2 A
Q3 C
Q4 C
Q5 D
Q6 B
Q7 D

Multiple Choice Working:

Q1. $\frac{\pi}{4} \times \frac{\pi}{3} = \frac{\pi^2}{12}$ C

Q2. $7(-\alpha)^3 + 9(-\alpha)^2 - 5\alpha(-\alpha) = 0$

$-7\alpha^3 + 9\alpha^2 + 5\alpha^2 = 0$

$-\alpha^2(7\alpha - 14) = 0$

$\therefore \alpha = 2 \qquad [\alpha \neq 0]$ A

Q3. $\dot{x} = 2 \cdot 5 - 2 \cdot 5 \sin(0 \cdot 5t)$

$\therefore \dot{x}_{min} = 0$ C

Q4. $f(x) = \sin x + C$

$0 = 1 + C \Rightarrow C = -1$

$\therefore f(x) = \sin x - 1$ C

Q5. A: $\frac{1}{\cot\frac{\pi}{5}} = \tan\frac{\pi}{5}$

B: $\cot\left(\frac{3\pi}{10}\right) = \tan\left(\frac{\pi}{2} - \frac{3\pi}{10}\right) = \tan\left(\frac{\pi}{5}\right)$

C: $\frac{2\tan\left(\frac{\pi}{10}\right)}{1 - \tan^2\left(\frac{\pi}{10}\right)} = \tan\left(2 \times \frac{\pi}{10}\right) = \tan\left(\frac{\pi}{5}\right)$ D

$\Rightarrow$ could have also just put it into a calculator..

Q6. $\dot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$

$= \frac{d}{dx}\left(\frac{1}{2} \cdot \frac{4}{1-x^2}\right)$

$= \frac{d}{dx}\left(2(1-x^2)^{-1}\right)$

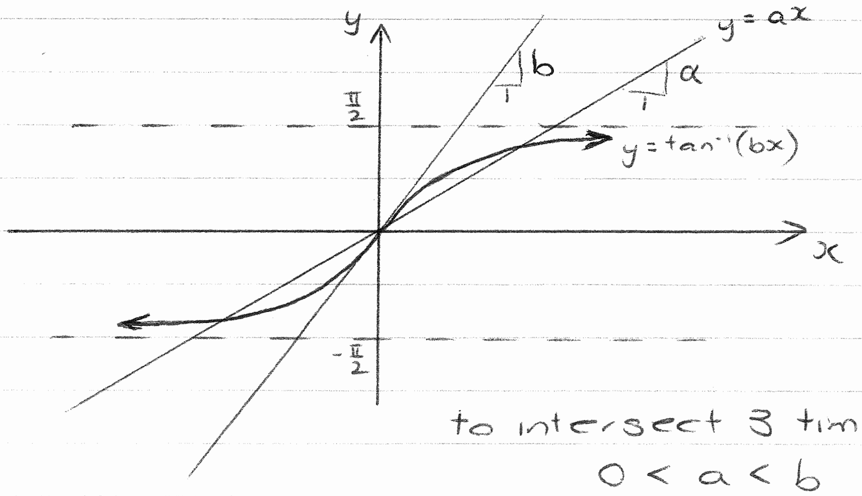
$= -2(1-x^2)^{-2}\, x - 2x$

$= \dfrac{4x}{(1-x^2)^2}$

B

Q7. $y = ax \rightarrow \frac{dy}{dx} = a \rightarrow m = a$

$y = \tan^{-1}(bx) \rightarrow \begin{cases} \frac{dy}{dx} = \frac{b}{1+b^2x^2} \\ \text{when } x = 0, \ m_T = b \end{cases}$



to intersect 3 times

$0 < a < b$

OR $b < a < 0$

D

QUESTION 8:

a) $-1 \leq \frac{x}{3} \leq 1$ $\therefore$ Domain: $-3 \leq x \leq 3$ ✓

b) $\int_0^{\sqrt{3}} \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_0^{\sqrt{3}}$ ✓

$= \sin^{-1}\frac{\sqrt{3}}{2} - \sin^{-1} 0$

$= \frac{\pi}{3}$ ✓

c) $\cos 15° = \cos(45° - 30°)$

$= \cos 45° \cos 30° + \sin 45° \sin 30°$ ✓

$\left.\begin{aligned} &= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2} \\ &= \frac{\sqrt{3}+1}{2\sqrt{2}} \text{ as required} \end{aligned}\right\}$ ✓

d) $2\sin^2\theta = \sin 2\theta$

$2\sin\theta(\sin\theta - \cos\theta) = 0$ ✓

$\therefore 2\sin\theta = 0$ or $\sin\theta - \cos\theta = 0$

$\sin\theta = 0$ $\quad$ $\tan\theta = 1$ $\quad$ $[\cos\theta \neq 0]$

$\therefore \theta = 0, \pi, 2\pi$ ✓ $\quad$ $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ ✓

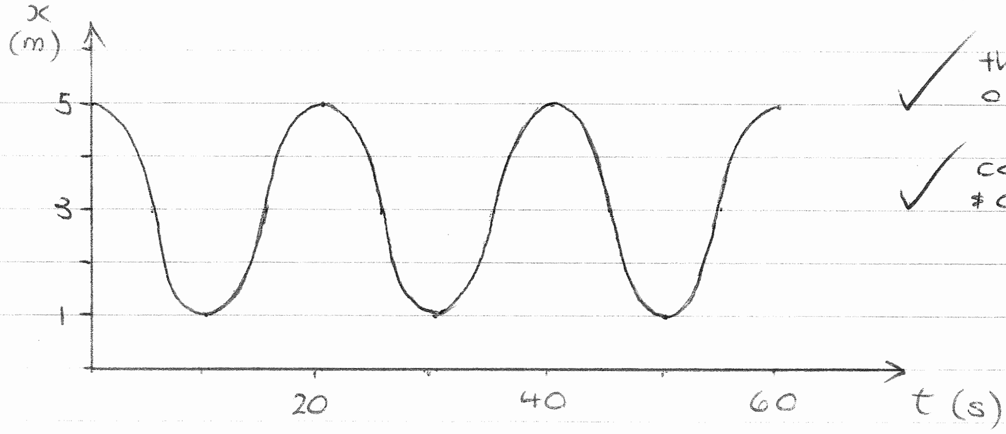
$\therefore \theta = 0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}$ or 2π

e) i) Amplitude = 2 metres ✓

Period $= 2\pi \div \frac{\pi}{10}$

$= 20$ seconds ✓

ii)



✓ three cycles of cos graph

✓ correct range & centre of motion

QUESTION 9:

a) When $t = 0$: $v = 0 \ ms^{-1}$
$x = \sqrt{6} \ m$

i) $\ddot{x} = 10x - 4x^3$
$= 10(\sqrt{6}) - 4(\sqrt{6})^3$
$= 10\sqrt{6} - 24\sqrt{6}$
$= -14\sqrt{6} \ ms^{-2}$ ✓ (exact value)

ii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 10x - 4x^3$

$\frac{1}{2}v^2 = 5x^2 - x^4 + c$ ✓

at $x = \sqrt{6}$, $v = 0$:
$0 = 5(\sqrt{6})^2 - (\sqrt{6})^4 + c$
$\therefore c = 6$
$\therefore v^2 = 10x^2 - 2x^4 + 12$ ✓

b) i) $P(1) = (1)^4 - (1)^3 - 3(1)^2 + 5(1) - 2$
$= 0$
$P(-2) = (-2)^4 - (-2)^3 - 3(-2)^2 + 5(-2) - 2$
$= 0$ ✓

ii) METHOD 1:
$x^4 - x^3 - 3x^2 + 5x - 2 = (x-1)(x+2)(x^2 + kx + 1)$
$= (x^2 + x - 2)(x^2 + kx + 1)$ ✓

Equating coeff's of x^3:
$-1 = k + 1$
$\therefore k = -2$ ✓

$\therefore P(x) = (x-1)(x+2)(x^2 - 2x + 1)$
$= (x-1)^3(x+2)$ ✓

OR

METHOD 2:

Let the zeroes be $1, -2, \alpha$ & β

Sum of roots: $1 + (-2) + \alpha + \beta = -\frac{(-1)}{1}$

$$\alpha + \beta = 2 \quad \text{—} ①$$

Product of roots: $1 \times (-2) \times \alpha \times \beta = \frac{-2}{1}$

$$\alpha\beta = 1 \quad \text{—} ②$$

✓ either

Rearrange ①: $\beta = 2 - \alpha$ ①*

Sub ①* into ②: $\alpha(2-\alpha) = 1$

$$\alpha^2 - 2\alpha + 1 = 0$$

$$(\alpha - 1)^2 = 0$$

$$\therefore \alpha = 1 \quad ✓$$

Sub into ①*: $\beta = 2 - 1$

$$= 1$$

$$\therefore P(x) = (x-1)^3(x+2) \quad ✓$$

OR

METHOD 3:

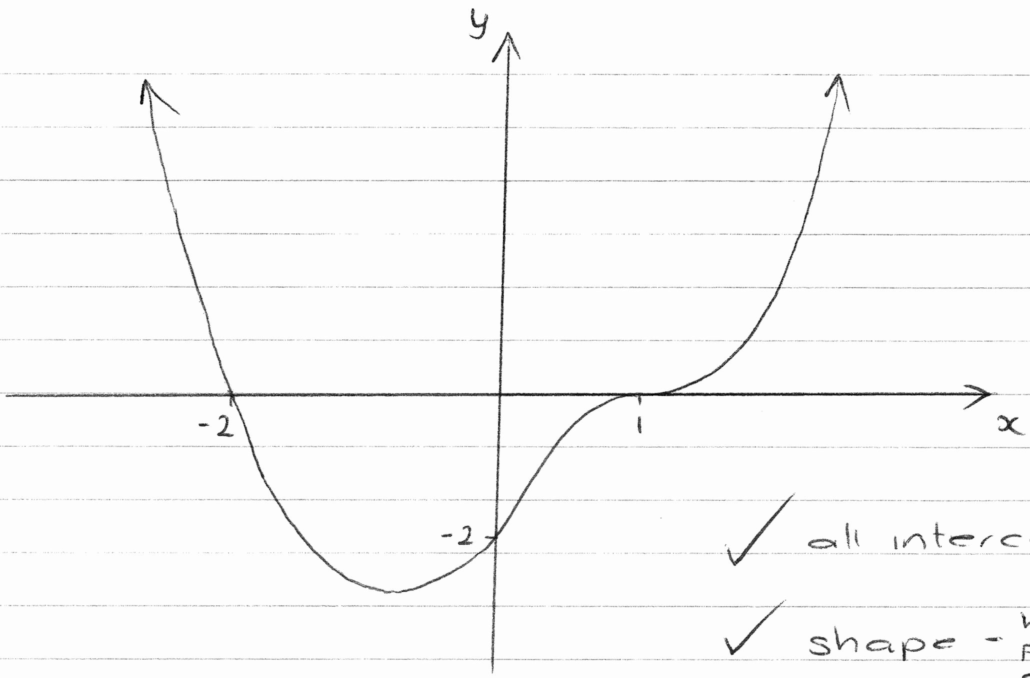
$$(x-1)(x+2) = x^2 + x - 2$$

$$\begin{array}{r|l}
 & x^2 - 2x + 1 \quad ✓ \\
\hline
x^2 + x - 2 \; ✓ & x^4 - x^3 - 3x^2 + 5x - 2 \\
 & \underline{x^4 + x^3 - 2x^2} \\
 & -2x^3 - x^2 + 5x \\
 & \underline{-2x^3 - 2x^2 + 4x} \\
 & x^2 + x - 2 \\
 & \underline{x^2 + x - 2} \\
 & 0
\end{array}$$

$$\therefore P(x) = (x-1)(x+2)(x^2 - 2x + 1)$$

$$= (x-1)^3(x+2) \quad ✓$$

iii)



✓ all intercepts

✓ shape - horizontal point of inflexion at 1 must be clear

c) i) $\frac{d}{dx}\left(x\sin^{-1}x + \sqrt{1-x^2}\right)$

$= \sin^{-1}x \times 1 + x \times \frac{1}{\sqrt{1-x^2}} + \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times -2x$ ✓

$= \sin^{-1}x + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$ (for either)

$= \sin^{-1}x$ ✓

ii) $\int_0^1 \sin^{-1}x \, dx = \left[x\sin^{-1}x + \sqrt{1-x^2}\right]_0^1$

$= 1 \times \frac{\pi}{2} + 0 - (0 + 1)$

$= \frac{\pi}{2} - 1$ ✓

QUESTION 10:

a) $4\cos^3 x - 3\cos x + 2\cos x = 0$

$$4\cos^3 x - \cos x = 0$$

$$\cos x\,(4\cos^2 x - 1) = 0$$

$$\cos x(2\cos x - 1)(2\cos x + 1) = 0$$

$$\therefore \cos x = 0,\ \tfrac{1}{2} \text{ or } -\tfrac{1}{2}$$

General solutions:

$$\cos x = 0 \rightarrow x = 2n\pi \pm \frac{\pi}{2} \quad (n \in \mathbb{Z})$$

$$\left.\begin{aligned} \cos x = \tfrac{1}{2} &\rightarrow x = 2n\pi \pm \frac{\pi}{3} \quad (n \in \mathbb{Z}) \\ \cos x = -\tfrac{1}{2} &\rightarrow x = 2n\pi \pm \frac{2\pi}{3} \quad (n \in \mathbb{Z}) \end{aligned}\right\}$$

[Alternatively:

$$\cos x = 0 \rightarrow x = \frac{\pi}{2} + n\pi$$

$$\cos x = \pm\tfrac{1}{2} \rightarrow x = n\pi \pm \frac{\pi}{3}$$]

b) i) $2\sin 3t - 2\sqrt{3}\cos 3t = R\sin(3t - \alpha)$

$$= R\sin 3t\cos\alpha - R\cos 3t\sin\alpha$$

$$\left.\begin{aligned} R\cos\alpha &= 2 \\ R\sin\alpha &= -2\sqrt{3} \end{aligned}\right\}$$

$R^2 = 2^2 + (2\sqrt{3})^2$ & $\tan\alpha = \frac{2\sqrt{3}}{2}$

$= 16$ $\qquad = \sqrt{3}$

$\therefore R = 4$ $\qquad \therefore \alpha = \frac{\pi}{3}$

$$\therefore x = 4\sin\left(3t - \frac{\pi}{3}\right)$$

ii) $\dot{x} = 12\cos\left(3t - \frac{\pi}{3}\right)$

when $t = 0$: $x = 4\sin\left(-\frac{\pi}{3}\right)$

$$= -2\sqrt{3}\text{ m}$$

$$\dot{x} = 12\cos\left(-\frac{\pi}{3}\right)$$

$$= 6 \text{ m/s}$$

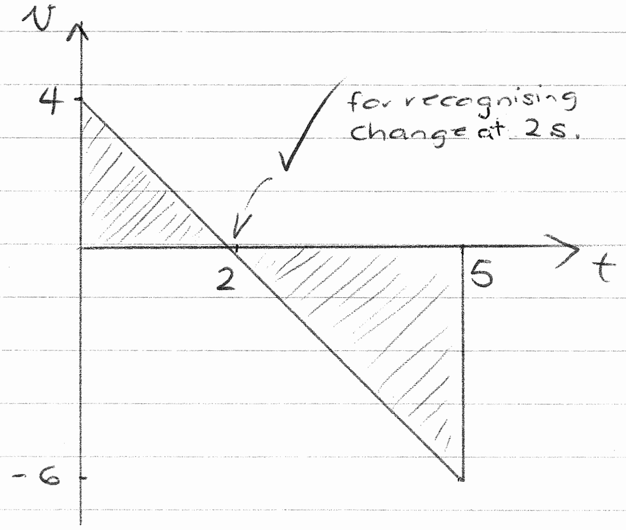
iii) since starts at $x = -2\sqrt{3}$ m ...
first time 2m from origin $\rightarrow$ $x = -2$ m

$$-2 = 4\sin\left(3t - \frac{\pi}{3}\right)$$
$$\sin\left(3t - \frac{\pi}{3}\right) = -\frac{1}{2}$$

$$3t - \frac{\pi}{3} = -\frac{\pi}{6}, \ \frac{7\pi}{6}, \ \ldots$$
$$t = \frac{\pi}{18}, \ \frac{\pi}{2}, \ \ldots$$

$\therefore$ the first time is at $t = \frac{\pi}{18}$ s.

c)



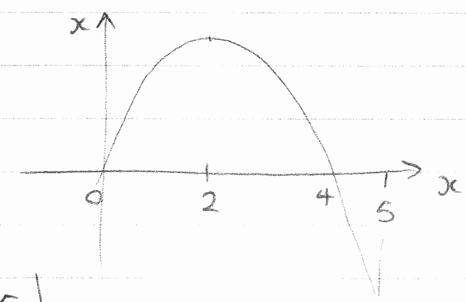
for recognising change at 2s.

distance travelled = total area

$$= \frac{1}{2} \times 2 \times 4 + \frac{1}{2} \times 3 \times 6$$
$$= 4 + 9$$
$$= 13 \text{ m}$$

Alternatively:

$$\dot{x} = 4 - 2t$$
$$x = 4t - t^2 + C$$
$$= t(4 - t) + C$$



Total distance $= 2 \times x_2 + |x_5|$

$$= 2(8 - 4) + |20 - 25|$$
$$= 8 + 5$$
$$= 13 \text{ m.}$$

QUESTION 11:

a) i) $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$

$= \dfrac{-\frac{b}{a}}{1 - \frac{c}{a}}$ ✓ ✓

$= \dfrac{b}{c-a}$

ii) $\tan^2(\alpha-\beta) = \left(\dfrac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}\right)^2$

$= \dfrac{\tan^2\alpha - 2\tan\alpha\tan\beta + \tan^2\beta}{(1+\tan\alpha\tan\beta)^2}$

$= \dfrac{(\tan\alpha + \tan\beta)^2 - 4\tan\alpha\tan\beta}{(1+\tan\alpha\tan\beta)^2}$ ✓

$= \dfrac{\left(-\frac{b}{a}\right)^2 - 4\left(\frac{c}{a}\right)}{\left(1+\frac{c}{a}\right)^2}$ ✓

$= \dfrac{\frac{b^2}{a^2} - \frac{4c}{a}}{\left(1+\frac{c}{a}\right)^2} \times \dfrac{a^2}{a^2}$

$= \dfrac{b^2 - 4ac}{(a+c)^2}$

b) i) $t = \dfrac{x}{V\cos\theta}$

$$y = V \times \frac{x}{V\cos\theta} \times \sin\theta - \frac{g}{2} \times \frac{x^2}{V^2\cos^2\theta} + \frac{V^2\sin^2\theta}{g}$$

$$= x\tan\theta - \frac{gx^2\sec^2\theta}{2V^2} + \frac{V^2\sin^2\theta}{g} \quad \text{as required}$$

ii) The keys will hit the ground when $y = 0$:

METHOD 1:

$$0 = Vt\sin\theta - \frac{gt^2}{2} + \frac{V^2\sin^2\theta}{g}$$

$a = \dfrac{g}{2}$

$b = V\sin\theta$

$c = \dfrac{V^2\sin^2\theta}{g}$

$$t = \frac{-V\sin\theta \pm \sqrt{V^2\sin^2\theta - 4 \times -\frac{g}{2} \times \frac{V^2\sin^2\theta}{g}}}{2 \times -\frac{g}{2}}$$

$$= \frac{-V\sin\theta \pm V\sin\theta\sqrt{3}}{-g}$$

$$= \frac{V\sin\theta(1+\sqrt{3})}{g} \quad (\text{since } t > 0)$$

$$x = \frac{V\cos\theta \times V\sin\theta(1+\sqrt{3})}{g}$$

$$= \frac{V^2(1+\sqrt{3})\sin 2\theta}{2g} \quad \text{as required.}$$

METHOD 2:

$$0 = x\tan\theta - \frac{g x^2 \sec^2\theta}{2V^2} + \frac{V^2\sin^2\theta}{g}$$ ✓

$$a = -\frac{g\sec^2\theta}{2V^2}$$

$$b = \tan\theta$$

$$c = \frac{V^2\sin^2\theta}{g}$$

$$x = \frac{-\tan\theta \pm \sqrt{\tan^2\theta - 4 \times -\frac{g\sec^2\theta}{2V^2} \times \frac{V^2\sin^2\theta}{g}}}{2 \times -\frac{g\sec^2\theta}{2V^2}}$$ ✓

$$= \frac{(-\tan\theta \pm \tan\theta\sqrt{3}) \times V^2}{-g\sec^2\theta}$$

$$= \frac{V^2\tan\theta(1+\sqrt{3})\cos^2\theta}{g} \quad (\text{since } x > 0)$$ ✓

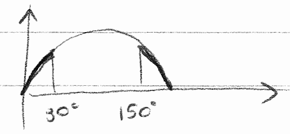
$$= \frac{V^2\sin\theta\cos\theta(1+\sqrt{3})}{g} \quad \left(\text{since } \tan\theta = \frac{\sin\theta}{\cos\theta}\right)$$

$$= \frac{V^2(1+\sqrt{3})\sin 2\theta}{2g} \quad \text{as required} \quad \left(\text{since } \sin\theta\cos\theta = \frac{\sin 2\theta}{2}\right)$$

iii) * Near side of moat:

$$0 < \frac{v^2(1+\sqrt{3})\sin 2\theta}{2g} < \frac{v^2(1+\sqrt{3})}{4g}$$ ✓

$$0 < \sin 2\theta < \frac{1}{2} \quad \text{for} \quad 0^\circ < 2\theta < 180^\circ$$



$$0^\circ < 2\theta < 30^\circ \quad \text{or} \quad 150^\circ < 2\theta < 180^\circ$$
$$0 < \theta < 15^\circ \qquad 75^\circ < \theta < 90^\circ$$ ✓

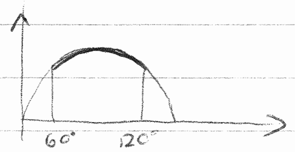
* Far side of moat:

$$\frac{v^2(1+\sqrt{3})\sin 2\theta}{2g} > \frac{v^2(1+\sqrt{3})}{4g} + \frac{v^2}{2g}$$

$$\frac{2v^2(1+\sqrt{3})\sin 2\theta}{4g} > \frac{v^2(3+\sqrt{3})}{4g}$$ ✓

$$\sin 2\theta > \frac{\sqrt{3}\cancel{(\sqrt{3}+1)}}{2\cancel{(1+\sqrt{3})}}$$

$$\sin 2\theta > \frac{\sqrt{3}}{2} \quad \text{for} \quad 0^\circ < 2\theta < 180^\circ$$



$$60^\circ < 2\theta < 120^\circ$$
$$30^\circ < \theta < 60^\circ$$ ✓

$$\therefore 0^\circ < \theta < 15^\circ, \ 30^\circ < \theta < 60^\circ \ \text{or} \ 75^\circ < \theta < 90^\circ$$